Inverse bremsstrahlung in strong radiation fields at low temperatures

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The process of above threshold ionization by plane polarized intense radiation fields has enabled cold ionized plasmas to be created. The final temperature depends on the heating process during the laser pulse. Collisional or inverse bremsstrahlung is a major effect at high density. It is argued that under these extreme conditions the conventional treatments of the Coulomb logarithm for electron-ion collisions is inadequate. Using a simple impact, or classical, approximation, corrected expressions and simple approximations are derived for this heating rate.

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I. INTRODUCTION

The impact approximation, originally misnamed the classical theory, was developed independently by the present author [1] and Bunkin, Kazakov, and Federov [2] over 20 years ago. The absorption in an electron–heavy-particle collision from the incident electromagnetic wave to thermal energy is identified with the reorientation of the quiver into translation motion in an elastic collision. The model is thus extremely simple and involves only averages over the dynamics of representative collisions. It is therefore particularly suited for direct calculation of the absorption rate, provided the electron distribution function is known. It has been shown by several authors [3–5] that this approach is asymptotically related by a saddle point integration to direct methods using either quantum or classical mechanics.

The calculation of absorption is based on the momentum transfer cross section for electron scattering. In a plasma this is dominated by long-range small-angle deflections. The cross section can be conveniently written in terms of the inner $b_{\min}$ and outer $b_{\max}$ impact parameters

$$\sigma_d = 4\pi b_{\min}^2 \ln[\{1+(b_{\max}/b_{\min})^2\}^{1/2}] . \qquad (1)$$

The inner is either the Landau parameter Ze^2/mv^2 for classical or the electron wavelength h/mv for quantal collisions, the appropriate value being the larger: v is the electron velocity, e and m its charge and mass, Z the ion fractional charge, and h Planck's constant. We shall concentrate on classical collisions in this paper, although the generalization will be considered. The outer impact parameter is determined by the requirement that the collision be completed in a time short compared to the period of the field, i.e., $b_{\max} \simeq v/\omega$, ω being the angular frequency of the field. If this condition is not established the electron scattering is averaged over the phase of the field and no net absorption results. This functional form is well established in alternative calculations and was discussed in some detail in Ref. [5]. Unfortunately alternative methods of calculation are all perturbative in some form. As a result it is only possible to calculate one of the cutoffs properly, the other (often the inner) being introduced in an *ad hoc* fashion. Fortunately the outer only appears as a logarithmic ratio in the cross section and this uncertainty as to the absolute value of $b_{\max}$ usually results in little error. The momentum transfer cross section is generally used for the calculation of transport properties, in which case $b_{\max}$ is the Debye length λ_D. Since Debye screening theory holds if $\lambda_D \gg r_0$, the interparticle separation length, and the collision approximation is $r_0 \gg b_{\min}$, the transport theory is only valid if $b_{\max} \gg b_{\min}$. As a result the "Coulomb" logarithm may be simplified to its usual form

$$\ln\Lambda \simeq \ln[b_{\max}/b_{\min}] . \qquad (2)$$

In the case of absorption, however, the electromagnetic wave must propagate in the plasma, so that $\omega > \omega_P$, the plasma frequency. Consequently, if the field is weak $v \sim v_T$, the thermal speed $b_{\max} < \lambda_D$, since $\lambda_D \sim v_T/\omega_P$, and only a limited range of impact parameters gives rise to absorption. Absorption is therefore more purely two body than other collisional processes. In particular, if the electron temperature is low, $b_{\max} < b_{\min}$ and the approximation (2) for the Coulomb logarithm will give rise to negative absorption. In fact, using the correct form (1) we see that $\sigma_d \simeq 2\pi b_{\max}^2$, as may be expected physically. However, calculation in this regime will be unreliable as the outer impact parameter is only known to an arbitrary factor of order unity, where the square will appear as a multiplying factor in the final result.

At the other extreme when the field is strong, $v \sim u_0$, the amplitude of the quiver velocity ($u_0 \gg v_T$), and $b_{\max} \sim u_0/\omega$, the amplitude of the quiver displacement. In this limit the electrons oscillate as a fluid continuum about the ions. Since $\omega > \omega_P$ the electron fluid cannot respond to the induced electric field fluctuation and screen it over distances greater than the Debye length as in the weak field and static cases. Consequently, the electrons can respond at ion separations (or impact parameters) $\sim u_0/\omega$, even if this is significantly greater than λ_D. Thus again in this limit we obtain $b_{\max} \sim v/\omega$, as before. Clearly, if $u_0/\omega \gg \lambda_D$ this may result in a substantial increase in the absorption rate. In this limit the electron speed varies from v_T to u_0. At the lower limit, the

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Coulomb logarithm may again be small, although the value at the upper limit is large. Consequently, the exact value of the logarithm (1) must be used instead of the approximation (2) if significant error is to be avoided. This is especially severe as the correct contribution from the region $v \sim 0$ is small as the term $\sigma_d/u \sim u^5 \rightarrow 0$, whereas the approximate form gives $\sigma_d/u \sim 1/u \ln(u) \rightarrow -\infty$.

In the past this has caused little problem and the form (2) has always been used to calculate absorption. This has been satisfactory as the temperature has been relatively high and the density close to critical ($\omega \sim \omega_P$). However, the advent of multiphoton ionization with ultrashort pulse lasers has given rise to very low temperature, highly ionized plasma. As a result the correct form (1) must be used. In this paper we will derive modified absorption formulas for this regime. The development of x-ray laser action during recombination following above threshold ionization (ATI) by short pulse laser breakdown is critically dependent on the temperature at the end of the pulse. At the relatively high pressure involved, inverse bremsstrahlung absorption may be one of the limiting factors involved in assessing the viability of this approach.

Although the duration of the pulse is short, it encompasses many periods of the wave. Since each impact has a duration less than the period and has a range, which will vary over the quiver oscillation, large compared to the inner impact parameter, the electron is scattered by many long-range small-angle collisions in a quasicontinuous fashion. As a result, although the net deflection through the period may be small and the effective collision time for 90° scattering large compared to the pulse duration, appreciable energy gain occurs as a significant fraction of the large quiver energy. The impact approximation being essentially phenomenological treats simultaneous multiple collisions on an equal footing with binary, the only proviso being that the "coherence time" of each individual scattering be short compared to the period of the electromagnetic wave. In consequence there is a direct relationship with the collective plasma calculation methods [6].

The rate of energy absorption depends on the details of the electron velocity distribution. In much early work it was assumed that electron-electron collisions were sufficiently fast that a Maxwell-Boltzmann distribution was maintained. However, Langdon [7] showed that if the fields were moderately strong (with the quiver speed approximately the thermal speed) the thermal distribution was not sustained and a self-similar form developed. In very strong fields, Jones and Lee [8] showed that the distribution reverted to a form of Maxwellian. As we are interested only in the strong field case at low temperature, we consider only a Maxwell distribution.

The low temperatures required for x-ray laser action in ATI are found only if the radiation is plane polarized. We therefore restrict our study to this case, although the extension to a different polarization form is straightforward. The emphasis of the calculation is towards high fields where the quiver energy is much larger than the thermal and the results using a Maxwell Boltzmann velocity distribution can only be considered to be valid in this condition. The approach is therefore to use asymptotic methods. Two methods of calculation are used. The first direct route involves the exact calculations of the absorption rate for the general cross section Eq. (1). The result involves a complex sum that can be performed numerically with some difficulty. The alternative approach is to use an asymptotic approximation that yields a simple integral. The relationship between the two methods is explored and used to derive a general high field solution. The use of the cutoffs in Eq. (1) introduces a further restriction on the range of the solution, namely, that total velocity is sufficiently large that $b_{\max} \gg b_{\min}$, or $\max(v_T, u_0) \gg [Ze^2\omega/m]^{1/3}$.

This model can be related to the standard theory of single photon bremsstrahlung emission from a hydrogenic ion developed by Sommerfeld [9] and by Menzel and Pekeris [10], which is the reciprocal process related by detailed balance in the weak field ($u_0 \ll v_T$) limit. The limiting cases were identified by Elwert [11] and Oster [12], who showed that the nature of the solution depended on the "principal quantum numbers" of the electron before $\eta_1 = Ze^2/\hbar v_1$ ($\hbar = h/2\pi$) and after $\eta_2 = Ze^2/\hbar v_2$ the collision and the threshold parameter $x = \hbar\omega/\frac{1}{2}mv_1^2$. For $x \ll 1$ and $|\eta_1 - \eta_2| \ll 1$ the logarithmic form of Eq. (1) is recovered, corresponding to scatterings for which the photon energy is sufficiently small that the path is nearly a straight line and the collision is completed in the period of the wave so that the impact is not modified by the field: this is the impact approximation and $\eta_1 \approx \eta_2$ since the process involves only a single photon. This condition can also be written $(2v/\omega)/(\hbar/mv) \gg 1$, i.e., the quantum form of $b_{\max} \gg b_{\min}$. For $\eta_1, \eta_2 \ll 1$ the Born (quantum) approximation is recovered and for $\eta_1, \eta_2 \gg 1/x$ the classical. Combining the latter conditions $\eta_1 x \gg 1$ we obtain $(2v/\omega)/(Ze^2/mv^2) \gg 1$, or $b_{\max} \gg b_{\min}$ once again. Thus within this theory the condition $b_{\max} \gg b_{\min}$, for which the impact model is valid, is seen to be associated with small photon energy. When this condition is violated ($b_{\max} \lesssim b_{\min}$), the parameter x is not in general small.

A very useful summary of the approximations and their regimes has been given by Brussaard and van de Hulst [13]. In this paper we limit consideration to nonrelativistic electrons, although the model is capable of generalization to include relativistic effects.

II. THE IMPACT APPROXIMATION

The impact approximation starts from a very simple dynamic result, namely, that in a collision where the quiver velocity is instantaneously $\mathbf{u}$ and the initial and final total velocities are $\mathbf{v}$ and $\mathbf{v}'$, the gain in the translational or thermal energy is $m\mathbf{u}\cdot(\mathbf{v}-\mathbf{v}')$. From this result we may obtain the energy gain in an average collision by averaging over the phase of the quiver motion (electromagnetic wave), the cross section, and the velocity distribution of the electrons. Averaging this result over the angular distribution of the cross section we obtain the energy gain from electrons of thermal velocity $\mathbf{v}_T$ and quiver velocity $\mathbf{u}$ per unit time [1,2]

$$R = mnv\mathbf{u}\cdot\mathbf{v}\sigma_d(v) , \tag{3}$$

where the total velocity $\mathbf{v}=\mathbf{u}+\mathbf{v}_T$ and the momentum transfer cross section for classical collisions is given by (1)

$$\sigma_d=2\pi\left[\frac{Ze^2}{m}\right]^2\frac{1}{v^4}\ln\left\{1+\frac{v^6}{(Ze^2\omega/m)^2}\right\} . \qquad (1')$$

We may now proceed in two alternative ways depending on the order in which the averages over the field and the distribution are performed. The first of these is complex, but yields an accurate theory, while the second gives simple but approximate results.

A. Accurate averaging

The accurate calculation of the energy absorption for a Maxwell-Boltzmann electron distribution was developed by the present author [14]. It was shown that the averaged energy absorption rate per electron per unit time (corrected for an omitted factor) was given by

$$\overline{R}=2n_i m(m/2\pi kT_e)^{1/2}\sum_{n=1}[2n/(2n+1)][1/(n!)^2]M(\tfrac{1}{2},(n+1),mu_0^2/2kT_e)\exp(-mu_0^2/2kT_e)$$
$$\times\int dv\,\sigma_d(v)v^2(mu_0v/2kT_e)^{2n}\exp(-mv^2/2kT_e) , \qquad (4)$$

where n_i is the ion density, T_e is the electron temperature, u_0 is the amplitude of the quiver velocity, and $M(a,b,x)$ is a confluent hypergeometric function. Equation (4) forms the starting point for much of our investigation.

For the case $b_{\max}\gg b_{\min}$, when the Coulomb logarithm may be written as in (2), Eq. (4) may be directly integrated to give

$$\overline{R}=\tfrac{2}{3}n_i m(m/2\pi kT_e)^{1/2}(mu_0^2/2kT_e)\times A[\ln\overline{\Delta}\mathcal{S}_1(x)+3\mathcal{S}_2(x)] , \qquad (5)$$

where A is given subsequently in Eq. (9) and

$$\overline{\Delta}=(2kT_e/m)^3/(Ze^2\omega/m)^2,\quad x=mu_0^2/2kT_e , \qquad (6)$$

$$\mathcal{S}_1(x)=3\sum_{n=1}^{\infty}[1/(2n+1)]x^{(n-1)}M(\tfrac{1}{2},(n+1),x)e^{-x} , \qquad (7)$$

$$\mathcal{S}_2(x)=3\sum_{n=1}^{\infty}[1/(2n+1)]\psi(n)x^{(n-1)}M(\tfrac{1}{2},(n+1),x)e^{-x} , \qquad (8)$$

and

$$\sigma_d=Av^{-4}\ln\Delta=2\pi\frac{Z^2e^4}{m^2}\frac{1}{v^4}\ln[v^6/(Ze^2\omega/m)^2] , \qquad (9)$$

where $\psi(n)$ is the digamma function. $\mathcal{S}_1(x)$ and $\mathcal{S}_2(x)$ take the analytic forms [14]

$$\mathcal{S}_1(x)={}_2F_2(\tfrac{3}{2},\tfrac{3}{2};\tfrac{5}{2},2;-x) ,$$
$$\mathcal{S}_1(x)+\gamma\mathcal{S}_2(x)=-\frac{d}{d\delta}{}_2F_2(\tfrac{3}{2},(\tfrac{3}{2}+\delta);\tfrac{5}{2},2;-x)|_{\delta=0} , \qquad (10)$$

where γ is Euler's constant ($=0.577\,215\,665$). The corrected asymptotic forms of $\mathcal{S}_1(x)$ and $\mathcal{S}_2(x)$ are obtained from these functions:

$$\mathcal{S}_1(x)\to3\pi^{-1/2}x^{-3/2}\left[\tfrac{1}{2}\ln(4x)+(\gamma/2+\ln2-1)-\pi^{-1}\sum_{n=1}\Gamma(\tfrac{3}{2}+n)\Gamma(\tfrac{1}{2}+n)x^{-n}/n!n\right] , \qquad (11)$$

$$\mathcal{S}_2(x)\to3\pi^{-1/2}x^{-3/2}\left[\tfrac{1}{4}(\ln4x)^2-(\gamma/2+\ln2-1)^2-\pi^2/24-1\right.$$
$$\left.-\pi^{-1}\sum_{n=1}\Gamma(\tfrac{3}{2}+n)\Gamma(\tfrac{1}{2}+n)(x^{-n}/n!n)\left[\ln(4x)+1/n-\sum_{m=1}^{n}\frac{2m}{(m^2-\frac{1}{4})}\right]\right] . \qquad (12)$$

Generalizing this result we obtain

$$\overline{R}=\tfrac{2}{3}n_i m(m/2\pi kT_e)^{1/2}(mu_0^2/2kT_e)A\mathcal{S}(\overline{\Delta},x) , \qquad (13)$$

where

$$\mathcal{S}(\overline{\Delta},x)=3\pi^{-1/2}x^{-3/2}\sum_{n=1}\frac{1}{(2n+1)}T_n(x)[1/(n-1)!]\times\int_0^{\infty}z^{(n-1)}\ln[1+\overline{\Delta}z^3]\times e^{-z}dz , \qquad (14)$$

$$T_n(x)=\sqrt{\pi}x^{(n+1/2)}M(\tfrac{1}{2},(n+1),x)e^{-x}/n! . \qquad (15)$$

For large x and $(x-n)\gg1$, the asymptotic form of $T_n(x)$ is derived in the Appendix. The complete term is shown in Fig. 1 and a good approximation is

$$T_n(x)\simeq\begin{cases}1/\sqrt{[1-n/x]}, & n<x-\sqrt{x}\\ 0 & \text{otherwise.}\end{cases} \qquad (16)$$

The maximum value of $T_n(x)$ is slightly less than that given above

$$T_{\max}(x)\simeq0.975x^{1/4} . \qquad (17)$$

It will be more convenient to work in terms of the function

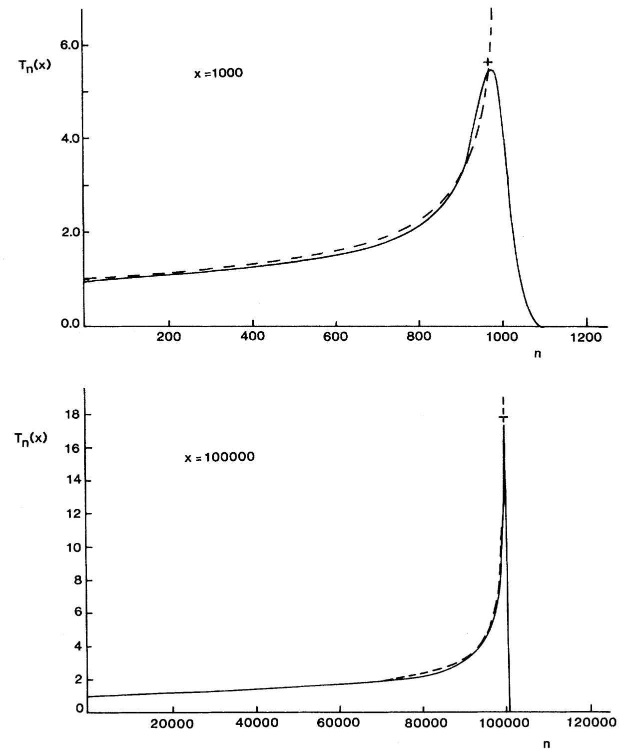

FIG. 1. Comparison of the function $T_n(x)$ evaluated directly (——) with the asymptotic result (· · · ·) for $x=10^3$ and 10^5 and varying index n. The point marks the approximate limit according to Eq. (17).

$$S(\bar{\Delta},x)=\tfrac{1}{3}\pi^{1/2}x^{3/2}\mathcal{S}(\bar{\Delta},x) \tag{18}$$

in terms of which

$$\bar{R}=\frac{2}{\pi}n_i m A\frac{1}{u_0}S(\bar{\Delta},x)\ . \tag{19}$$

The function S itself is also conveniently split into the terms already calculated S_1 and S_2 and a correction

$$\begin{aligned} S(\bar{\Delta},x)&=S_\infty(\bar{\Delta},x)+S'(\bar{\Delta},x)\\ &=S_1(x)\ln\bar{\Delta}+3S_2(x)+S'(\bar{\Delta},x)\ , \end{aligned} \tag{20}$$

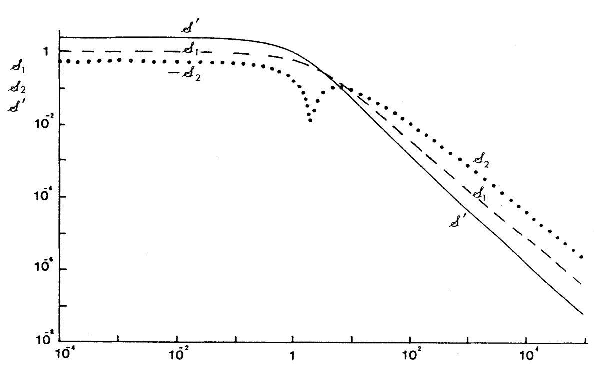

FIG. 2. Plot of the classical correction function $\mathcal{S}'(1,x)$(——) and the terms $\mathcal{S}_1(x)$(— — —) and $\mathcal{S}_2(x)$ (· · · ·).

where

$$S'(\bar{\Delta},x)=\sum_{n=1}\frac{1}{(2n+1)}T_n(x)\frac{1}{(n-1)!}\times\int_0^\infty \ln[1+1/\bar{\Delta}z^3]z^{(n-1)}e^{-z}dz\ . \tag{21}$$

Since the integrals have values $\sim 1/n^3$, this usually forms a rapidly convergent series. For small n the integral is easily evaluated by Gauss-Laguerre integration and for large n by the method of steepest descent. Figure 2 shows a plot of $\mathcal{S}'$ for $\bar{\Delta}=1$ obtained by direct summation. It can be seen that the correction is generally quite small, unless x is small where the approximations are involved.

In the high field limit for which a Maxwell-Boltzmann distribution is appropriate, the sum (21) is easily obtained by noting that $T_n(x)\simeq 1$. Figure 3 shows a plot of this function, compared with an approximation to be discussed later.

B. Asymptotic averaging

A simpler method of averaging that yields approximate results in the limit x large is to consider the behavior of particles of fixed thermal velocity v_T for which we obtain [1] that the energy gain for an electron per unit time with quiver speed u

$$R\simeq\begin{cases}0, & v_T>|u|\\ \dfrac{mn_i A\ln\langle\Delta\rangle}{|u|}, & v_T<|u|,\end{cases} \tag{22}$$

where $\langle\Delta\rangle$ is an appropriate average of Δ. The zero for $|u|<v_T$, which is a consequence of the averaging of Δ, is only important for low fields $u_0\ll v_T$ and is discussed elsewhere [5]. Averaging over the period of the wave

$$\bar{R}=\frac{2}{\pi}mn_i A\frac{1}{u_0}\int_0^{\pi/2}\frac{\ln[\langle\Delta(u_0\sin\theta)\rangle]d\theta}{\sin\theta}\ . \tag{23}$$

This results is exact in the limit $v_T\to 0$, i.e., $u_0\gg v_T$, the case of interest in ATI, and can form the basis of an approximation for finite temperature. Let us assume that $\langle\Delta\rangle$ is replaced by an average over the quiver oscillation; then

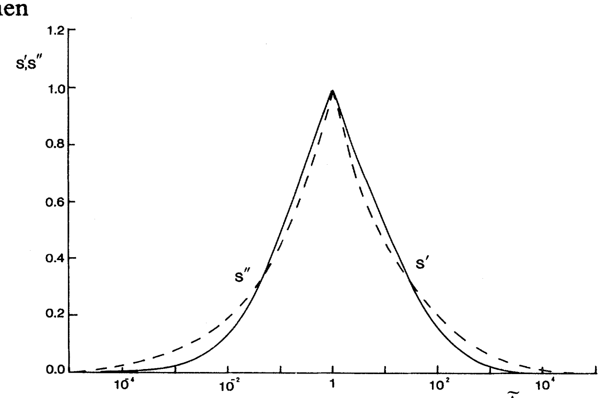

FIG. 3. Plot of the classical correction function $S'(\bar{\Delta},x)$ for large x compared with its approximation (— — —) given in Eq. (52). For $\Delta<1$ the function $S''(\Delta,x)$ is shown.

$$\bar{R}=\frac{2}{\pi}mn_i A\frac{1}{u_0}\ln\bar{\Delta}\ln\left[\frac{2u_0}{v_T}\right],\quad u_0\gg v_T \tag{24}$$

and averaging over the thermal distribution

$$\bar{R}=\frac{2}{\pi}mn_i A\frac{1}{u_0}\ln(\bar{\Delta})[\tfrac{1}{2}\ln(4x)+(\gamma/2+\ln2-1)]\ . \tag{24a}$$

A comparison with Eqs. (5) and (11) shows this to be identical provided we neglect the term associated with the variation of the logarithm. We shall show in the next section that (23) is the asymptotic limit of (5) as $x\to\infty$.

III. RELATIONSHIP BETWEEN THE TWO METHODS

It is clear from this result that the two approaches have a clear relationship in the limit $x\to\infty$. In particular, if we set $v_T=\sqrt{2kT_e/m}$ and $x=(u_0/v_T)^2$ then Eq. (23) may be written

$$\bar{R}=\frac{2}{\pi}mn_i A\frac{1}{u_0}\int_{\arcsin(x^{-1/2})}^{\pi/2}\ln\{1+\bar{\Delta}x^3\sin^6\theta\}\frac{d\theta}{\sin\theta}\ , \tag{25}$$

which we may compare with Eq. (5), with the integral replaced by its saddle point approximation and $T_n(x)$ by (16). If $n=x\sin^2\theta$ it is seen that the two are equivalent for large n, if the sum is replaced by an integral. As a further check we evaluate (25) for the case $\bar{\Delta}\gg1$, to yield for $x\gg1$

$$\bar{R}=\frac{2}{\pi}mn_i A\frac{1}{u_0}\left\{\ln(\bar{\Delta})\tfrac{1}{2}\ln(4x)+3\left[\tfrac{1}{4}[\ln(4x)]^2-\frac{\pi^2}{12}\right]\right\}\ . \tag{25a}$$

The relationship of this form with the asymptotic solution of (5) is evident. The origin of the difference between the two equations is easily identified as due to the inadequate treatment of the thermal velocity distribution in the approximation. This leads to two sources of error, which we can separate. The first is due to the replacement of the sum by an integral and the representation of the digamma term by its asymptotic form; the second contribution is from the neglected terms in the asymptotic approximation. The effect of the summation in the limit of large x can be calculated and shown to account for nearly all the difference:

$$\lim_{x\to\infty}\sum_{n=1}^{\infty}T_n(x)/(2n+1)-\int_{\arcsin(x^{-1/2})}^{\pi/2}d\theta/\sin\theta=\gamma/2+\ln2-1$$
$$\to\lim_{N\to\infty}\sum_{n=1}^{N}1/(2n+1)-\tfrac{1}{2}\int_1^N dn/n=\gamma/2+\ln2-1\ , \tag{26}$$

in agreement with (24a), and

$$\lim_{x\to\infty}\sum_{n=1}^{\infty}T_n(x)\psi(n)/(2n+1)-\int_{\arcsin(x^{-1/2})}^{\pi/2}\ln(x\sin^2\theta)d\theta/\sin\theta=-(\gamma/2+\ln2-1)^2-1+\pi^2/24=-0.589\,099\,362$$
$$\to\lim_{N\to\infty}\sum_{n=1}^{N}\psi(n)/(2n+1)-\tfrac{1}{2}\int_1^N\ln(n)dn/n=-(\gamma/2+\ln2-1)^2-2(1-\ln2)+0.024\,939\,149\,2$$
$$=-0.589\,099\,369\ . \tag{27}$$

More generally, if we define the integral

$$\tilde{S}(\bar{\Delta},x)=\int_{\arcsin(x^{-1/2})}^{\pi/2}\ln\{1+\bar{\Delta}x^3\sin^6\theta\}d\theta/\sin\theta \tag{28}$$

and proceed as before, we obtain for x sufficiently large (≥10)

$$S(\bar{\Delta},x)=\tilde{S}(\bar{\Delta},x)+[S'(\bar{\Delta},\infty)-\tilde{S}'(\bar{\Delta},\infty)]+(\gamma/2+\ln2-1)\ln\bar{\Delta}-1.767\,298\,088\ , \tag{29}$$

where

$$S'(\bar{\Delta},\infty)=\sum_{n=1}^{\infty}\left[\int_0^\infty\ln(1+1/\bar{\Delta}t^3)t^{(n-1)}\times e^{-t}dt\right]/(n-1)!(2n+1)\ , \tag{30}$$

$$\tilde{S}'(\bar{\Delta},\infty)=\int_0^{\pi/2}\ln(1+1/\bar{\Delta}t^3)dt/t$$
$$=\begin{cases}\pi^2/36+\frac{1}{12}[\ln\bar{\Delta}]^2-\frac{1}{6}\sum\limits_{m=1}\dfrac{(-1)^m\bar{\Delta}^m}{m^2}, & \bar{\Delta}<1\\[2ex] \frac{1}{6}\sum\limits_{m=1}\dfrac{(-1)^m\bar{\Delta}^{-m}}{m^2}, & \bar{\Delta}>1\ .\end{cases} \tag{31}$$

The terms $S'(\bar{\Delta},\infty)$ can be evaluated rapidly since the sum is reasonably convergent unless $\bar{\Delta}\ll1$, in which case its value is not required. The integrals are obtained by using either Gauss-Laguerre integration or the method of steepest descent. In fact, we give later a useful approximation based on (29).

This is a fortunate result as $\tilde{S}(\bar{\Delta},x)$ is much easier to evaluate than $S(\bar{\Delta},x)$. We only require results in the case $\bar{\Delta}\gg1$ or $\bar{\Delta}x^3\gg1$ when the argument of the Coulomb logarithm is large and theory based on cutoffs is satisfactory. The case $\bar{\Delta}\gg1$ is already covered by Eqs. (5) and (10) and by the approximations given earlier. The separate case $\bar{\Delta}x^3\gg1$ implies $x\gg1$, i.e., the high field case for which (25) may be used. For finite x Eq. (28) can be rapidly evaluated numerically, for example, by a midpoint Romberg method, but for large x suitable approximations will be developed.

IV. ZERO TEMPERATURE

In the limit $v_T\to0$, Eq. (28) is exact. Thus

$$\bar{R}=\frac{2}{\pi}mn_i A\frac{1}{u_0}\int_0^{\pi/2}\ln[1+\tilde{\Delta}\sin^6\theta]d\theta/\sin\theta\ , \quad (32)$$

where $\tilde{\Delta}=(mu_0^2)^3/(Ze^2\omega/m)^2$. The integral can be evaluated analytically in terms of a set of expansions.

(a) $\tilde{\Delta}<1$: In this case the argument of the Coulomb logarithm is small and the cutoff approximation is not valid. We include the result for completeness

$$S_0=\int_0^{\pi/2}\ln[1+\tilde{\Delta}\sin^6\theta]d\theta/\sin\theta$$

$$=\sum_{n=1}^{\infty}(-1)^{(n-1)}[\Gamma(\tfrac{1}{2})\Gamma(3n)/2n\Gamma(3n+\tfrac{1}{2})]\tilde{\Delta}^n\ . \quad (33)$$

An accurate approximation is

$$S_0\simeq 0.815\,06\ln[1+0.654\,34\tilde{\Delta}]\ . \quad (34)$$

(b) $\tilde{\Delta}>1$:

$$S_0=\int_0^{\theta_1}\ln[1+\tilde{\Delta}\sin^6\theta]d\theta/\sin\theta+\int_{\theta_1}^{\pi/2}\ln[\tilde{\Delta}\sin^6\theta]d\theta/\sin\theta+\int_{\theta_1}^{\pi/2}\ln[1+1/\tilde{\Delta}\sin^6\theta]d\theta/\sin\theta\ , \quad (35)$$

$$\int_0^{\theta_1}\ln[1+\tilde{\Delta}\sin^6\theta]d\theta/\sin\theta=\frac{1}{6}\left\{\frac{\pi^2}{12}+3\sum_{n=1}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})nn!}\tilde{\Delta}^{-n/6}[\ln 2-\beta(1+n/3)]\right\}, \quad (36)$$

$$\int_{\theta_1}^{\pi/2}\ln[\tilde{\Delta}\sin^6\theta]d\theta/\sin\theta=6\left\{\tfrac{1}{2}(\ln 2)^2+\tfrac{1}{6}\ln(\tilde{\Delta})\ln(t_1)-\tfrac{1}{2}[\ln(2t_1)]^2-\frac{1}{2}\left[\frac{\pi^2}{12}-\sum_{n=1}\frac{(-1)^n t_1^{-2n}}{n^2}\right]\right\}, \quad (37)$$

$$\int_{\theta_1}^{\pi/2}\ln[1+1/\tilde{\Delta}\sin^b\theta]d\theta/\sin\theta=\sum_{n=1}\frac{(-1)^{(n-1)}\mathcal{I}_{(6n+1)}}{n}\ , \quad (38)$$

where

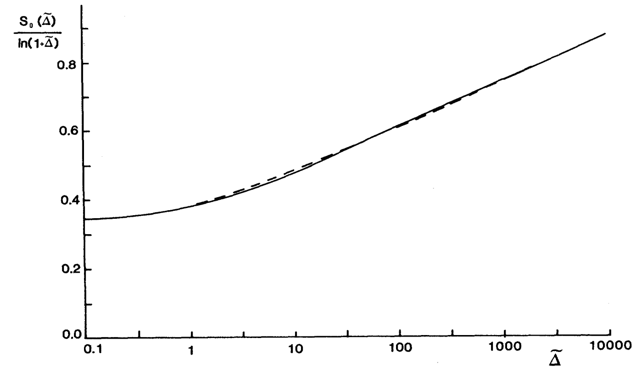

FIG. 4. Plot of the zero temperature absorption function plotted as $S_0(\tilde{\Delta})/\ln(1+\tilde{\Delta})$(——) compared with its approximation (— — —).

$$\mathcal{I}_{(6n+1)}=\frac{(6n-1)(6n-3)(6n-5)}{(6n)(6n-2)(6n-4)}\tilde{\Delta}^{-1}\mathcal{I}_{(6n-5)}+\frac{\cos\theta_1}{(6n)}\left\{1+\frac{(6n-1)}{(6n-2)}\sin^2\theta_1\times\left[1+\frac{(6n-3)}{(6n-4)}\sin^2\theta_1\right]\right\}, \quad (39)$$

$$\mathcal{I}_1=-\ln[\tan(\theta_1/2)]=\ln(t_1)\ , \quad (40)$$

$\beta(z)$ is the beta function

$$\beta(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(z+n)}\ , \quad (41)$$

and

$$\theta_1=\arcsin(1/\tilde{\Delta}^{1/6})\ ,\qquad t_1=\cot(\theta_1/2)=\tilde{\Delta}^{1/6}+\sqrt{[\tilde{\Delta}^{1/3}-1]}\ . \quad (42)$$

A satisfactory approximation for this case is

$$S_0=\begin{cases}0.969\,08\ln(1+0.527\,15\tilde{\Delta}), & 1\le\tilde{\Delta}\le 10\\ 3\{[\ln 2]^2+\frac{1}{3}\ln(\tilde{\Delta})\ln(t_1)-[\ln(t_1/2)]^2-\pi^2/12+t_1^{-2}\}+\pi^2/72 & \\ \quad+[\frac{15}{48}\tilde{\Delta}^{-1}\ln t_1+(\pi^2/72)\cos(\theta_1)(1+\frac{5}{4}\sin^2\theta_1+\frac{15}{8}\sin^4\theta_1)], & \tilde{\Delta}>10\ .\end{cases} \quad (43)$$

Figure 4 shows a plot of $S_0(\tilde{\Delta})$ compared with the approximations. For large $\tilde{\Delta}$ Eq. (35) reduces to

$$S_0=\tfrac{1}{12}(\ln[64\tilde{\Delta}])^2-2\pi^2/9\ . \quad (44)$$

V. FINITE TEMPERATURE

We may also carry out the integration of Eq. (28) for the case of finite temperature, although the results will clearly only give an approximation to the exact theory (5). Thus, for the case x large we obtain

$$\tilde{S}(\bar{\Delta},x)\simeq\begin{cases}3[\frac{1}{2}\ln(4x)]^2+\ln(\bar{\Delta})[\frac{1}{2}\ln(4x)]+\frac{1}{12}[\ln\bar{\Delta}]^2-\pi^2/4+\pi^2/72+\frac{1}{6}\left[\pi^2/12-\sum_{n=1}\frac{(-1)^{n-1}\bar{\Delta}^n}{n^2}\right], & \bar{\Delta}<1\\ 3[\frac{1}{2}\ln(4x)]^2+\ln(\bar{\Delta})[\frac{1}{2}\ln(4x)]-\pi^2/4+\frac{1}{6}\sum_{n=1}\frac{(-1)^{(n-1)}\bar{\Delta}^{-n}}{n^2}, & \bar{\Delta}>1.\end{cases} \quad (45)$$

In the case $\bar{\Delta}\simeq 0$ this yields (44), as expected, and for $\bar{\Delta}\gg 1$ an approximation to (11) and (12), which is in error by the terms investigated in (26) and (27). Alternatively we may use Eq. (45) to suggest an approximation for the correction $S'(\bar{\Delta},x)$ in the following form:

$$S(\bar{\Delta},x)=\begin{cases}S_\infty(\bar{\Delta},x)+S'(\bar{\Delta},x), & \bar{\Delta}>1\\ S_0(\bar{\Delta},x)-S''(\bar{\Delta},x), & \bar{\Delta}<1.\end{cases} \quad (46)$$

Calculating the value of $S(1,x)$ directly we find that to a reasonable degree of accuracy

$$\frac{S'(1,x)}{S''(1,x)}\simeq\frac{S'(1,\infty)}{S''(1,\infty)}\simeq\frac{0.9988}{1.044}\simeq 1$$

and that

$$S'(\bar{\Delta},x)\simeq[\ln(1+125/\bar{\Delta})/\ln(126)]S'(1,x)$$
$$\simeq 0.2045\ln(1+125/\bar{\Delta})S'(1,x) \quad (46a)$$

and

$$S''(\bar{\Delta},x)\simeq[\ln(1+75\bar{\Delta})/\ln(76)]S''(1,x)$$
$$\simeq 0.2284\ln(1+75\bar{\Delta})S''(1,x) \quad (46b)$$

to provide a simple approximation valid for $x\geq 2$ with reasonable accuracy if used with the forms for S_1 and S_2 given earlier and with S_0 from (43). Figure 3 shows the accuracy of these approximations.

VI. IMPACT APPROXIMATION FAILURE

There is one remaining problem arising when the impact approximation fails as $b_{\max}<b_{\min}$, i.e., $\bar{\Delta}<1$. In general this is only a problem in weak fields $u_0\ll v_T$ and as we noted in the Introduction at low temperature $\frac{1}{2}mv_T^2<\hbar\omega$. Thus we may consider the problem in the context of the single-photon absorption model. The related problem in emission was discussed by Oster [15], who showed that the change from classical to threshold behavior was rapid and that good accuracy could be obtained by a switch. Such an approach is also appropriate in this case.

In the standard theory of bremsstrahlung emission, the frequency dependence is generally described in terms of a Gaunt factor g. This term averaged over velocity $\bar{g}$ carries over directly into absorption. It was noted earlier [14] that in the low field limit $x\to 0$, the term $S_\infty(0)$ is identical to the Gaunt factor $\bar{g}$ apart from a small numerical factor associated with cutoff, namely,

$$S_\infty(0)=(2\pi/\sqrt{3})\bar{g}\simeq(2\pi/\sqrt{3})g((2kT/\gamma^*m)^{1/2}),$$

where $\ln\gamma^*=\gamma$. This result must hold quite generally even outside the range of the impact approximation. At threshold $v_T\to 0$, the Gaunt factor $\bar{g}\approx 1$. Hence a suitable form that satisfies this limiting condition is to replace $\ln\bar{\Delta}$ by

$$\ln\bar{\Delta}\Longrightarrow\max[\ln\bar{\Delta},5.359\,246].$$

A similar correction can be applied in the quantum case in the unlikely event of being required.

VII. THE QUANTUM LIMIT

If the electron wavelength exceeds the Landau parameter the scattering at small impact parameters in essentially quantum mechanical and described by the Born approximation. In this case the impact approximation is also an asymptotic limit valid when the energy absorbed per collision is much greater than the photon energy, i.e., $mv_0\gg\hbar\omega$ [5]. In this approximation the Coulomb logarithm appears naturally in the form (2) with

$$\frac{b_{\max}}{b_{\min}}\simeq\frac{2mv^2}{\hbar\omega}\gg 1\,. \quad (47)$$

The absorption coefficient is readily calculated in this limit if the collision can be described entirely within this approximation by a direct extension of the theory developed for classical collisions, but we require only the simpler limits $\bar{\beta}=(4kT/\hbar\omega)^2\gg 1$ and $\tilde{\beta}=\bar{\beta}x^2\gg 1$. Introducing the parameter S defined as before, we have, provided $\bar{\Delta}/\bar{\beta}>1$,

$$S(\bar{\beta},x)=\begin{cases}\ln(\bar{\beta})S_1(x)+2S_2(x), & kT\to\infty\\ S_0(\tilde{\beta}), & kT\to 0,\end{cases} \quad (48)$$

where S_1 and S_2 are given by (10) and (18) and

$$S_0(\tilde{\beta})=\tfrac{1}{8}\{\ln[16\tilde{\beta}]\}^2-\pi^2/8,\quad \tilde{\beta}\gg 1\,, \quad (49)$$

an additional term $\pi^2/24$ being included to account for the period when $\tilde{\beta}u_0\cos(\omega t)<1$ by analogy with the classical result (44).

For large fields $x\gg 1$ we may develop an approximation for $S(\bar{\beta},x)$ analogous to (45),

$$\tilde{S}(\bar{\beta},x)=\begin{cases}2[\frac{1}{2}\ln(4x)]^2+\ln(\bar{\beta})[\frac{1}{2}\ln(4x)]+\frac{1}{8}[\ln\bar{\beta}]^2-\pi^2/6+\frac{1}{4}\left\{\pi^2/6-\sum_{n=1}\frac{(-1)^{(n-1)}\bar{\beta}^n}{n^2}\right\}, & \bar{\beta}<1\\ 2[\frac{1}{2}\ln(4x)]^2+\ln(\bar{\beta})[\frac{1}{2}\ln(4x)]-\pi^2/6+\frac{1}{4}\sum_{n=1}\frac{(-1)^{(n-1)}\bar{\beta}^{-n}}{n^2}, & \bar{\beta}>1.\end{cases} \tag{50}$$

The small terms in large curly brackets represent components introduced by the use of the Coulomb logarithm in form (1) rather than (2) to allow for the invalid part of the integral where $\tilde{\beta}u^4$ is small. As before we may conveniently write

$$S(\bar{\beta},x)=\begin{cases}S_\infty(\bar{\beta},x)+S'(\bar{\beta},x), & \bar{\beta}>1\\ S_0(\bar{\beta},x)-S''(\bar{\beta},x), & \bar{\beta}<1,\end{cases} \tag{51}$$

where a reasonable matching function is

$$S'(\bar{\beta},x)\simeq 0.1676\ln(1+100/\bar{\beta}) ,$$
$$S''(\bar{\beta},x)\simeq 0.1676\ln(1+100\bar{\beta}) . \tag{52}$$

More generally the transition from classical to quantum behavior is not straightforward as either u_0 or v_T increases, as the electron may exhibit both types of behavior within different parts of its cycle. Indeed within a cycle where the scattering is quantum at its fastest limit it will be classical at its slowest if v_T is sufficiently small. In this case it is relatively easy to evaluate the integral form, but the thermal correction may be more difficult to obtain. A relatively simple approximation is formed as a correction to the classical results obtained earlier with the condition $\bar{\Delta}/\bar{\beta}<1<\tilde{\Delta}/\tilde{\beta}$:

$$S=S_{\text{class}}-S_{\text{corr}} , \tag{53}$$

where

$$S_{\text{corr}}=\sum_{n=1}[T_n(x)/(2n+1)]\int_{(\bar{\beta}/\bar{\Delta})}^{\infty}\ln(\bar{\Delta}t^3/\bar{\beta}t^2)t^{(n-1)}e^{-t}dt/(n-1)! \tag{54}$$

$$\simeq\int_{\arcsin(\tilde{\beta}/\tilde{\Delta})^{1/2}}^{\pi/2}\ln[\tilde{\Delta}/\tilde{\beta}\sin^2\theta]d\theta/\sin\theta \tag{55}$$

$$=-[\ln t']^2-2\ln(2\tilde{\Delta}^{1/2}/\tilde{\beta}^{1.2})\ln(t')-\pi^2/12+\sum_{n=1}\frac{(-1)^{(n-1)}t'^{2n}}{n^2} \tag{56}$$

$$\simeq\frac{1}{4}\{\ln[4\tilde{\Delta}/\tilde{\beta}]\}^2-\pi^2/12,\quad \tilde{\Delta}\gg\tilde{\beta} , \tag{57}$$

where $t'=\tilde{\beta}^{1/2}/[\tilde{\Delta}^{1/2}+(\tilde{\Delta}-\tilde{\beta})^{1/2}]$ and we have made use of the fact that in the quantum regime the Coulomb logarithm in the form (2) may be used with little error. Comparisons of Eqs. (45) and (50) show that this correction is appropriate at the transition to a full quantum calculation ($\bar{\Delta}/\bar{\beta}=1$) if the asymptotic forms are used, but comparing (14) and (48) reveals a small error associated with the terms $n\simeq 1$ in the full sum. Since these correspond to low velocity collisions, i.e., classical, the error in the use of (53) will be small unless $\bar{\Delta}/\bar{\beta}\simeq 1$ when a transition to (51) is appropriate. The purely classical condition corresponds to $\tilde{\Delta}/\tilde{\beta}<1$.

VIII. THE IMPACT APPROXIMATION FAILURE AT HIGH FIELDS

For completeness we finally consider one final case of relatively little importance at present, where the quiver energy is much larger than the photon energy, which is itself large compared to the thermal, i.e.,

$$mu_0^2\gg\hbar\omega\gg mv_T^2 . \tag{58}$$

The solution to this problem is given by the Born approximation and was derived some time ago [16,17]. Improving the accuracy of the logarithm arising from the cutoff of the asymptotic form of the Bessel function $J_n(z)$ (namely, $z=n$ instead of 1), the photon absorption cross section for the simultaneous absorption of n photons in the limit $v_T\to 0$ is

$$\sigma_n=\frac{2^5\pi\omega n_i Z^2e^3}{cE^3n}\ln(mu_0^2/n\hbar\omega) \tag{59}$$

and summing over n up to the classical limit, the rate is given by (19) with

$$S=\frac{1}{4}\ln(mu_0^2/\hbar\omega) . \tag{60}$$

This result differs from $S_0(\tilde{\beta})$ in Eq. (49) by a factor of 2. The latter is calculated outside the range of its validity, and should strictly be used only as a term in the approximation (51).

IX. DISCUSSION

In this paper we have developed a series of expressions for the energy increase during electron-ion collisions by inverse bremsstrahlung. The results form a continuous set of approximations that span the parameter space within which the photon energy is small, i.e., the radiation field classical. The calculations are based on the impact approximation for inverse bremsstrahlung, which has a simple phenomenological interpretation, but has also been showed to be an asymptotic limit of more sophisticated collision theory. The simplicity of this model in which individual collisions can be isolated and not considered within some form of ensemble averaging is of great advantage in formulating convenient solutions within different regions of parameter space. Using a general formulation developed earlier, the asymptotic form is shown to correspond to simple temporal averaging. The results have been obtained using an isotropic Maxwell-Boltzmann distribution, although it is well known that significant departures from this form occur. At intermediate field strengths $u_0 \sim v_T$ electron-electron collisions cannot maintain a thermal distribution against the changes induced by ion-electric collisions. This leads to preferential heating of slow electrons and narrower distribution with reduced absorption. This effect is well known following the work of Langdon [7]. However, we may argue that if the correct form of the Spitzer logarithm is used, as in Eq. (1), rather than a constant average value, as is conventional, this effect is reduced. The singularity in the cross section as $v \rightarrow 0$ is replaced by a zero and the heating of slow electrons reduced. The distribution may, depending on the value of $\overline{\Delta}$, more closely match a Maxwellian than is conventionally assumed.

At high fields ($u_0 \gg v_T$) electron-ion scattering essentially imparts a transverse velocity to the electrons. In consequence the distribution is determined by that of the scattering, i.e., Gaussian from the central limit theorem, in the transverse plane and Gaussian with a smaller random velocity in the field direction, as found by Jones and Lee [8]. In consequence a Maxwellian is a reasonable approximation; indeed in these conditions the absorption rate is essentially determined by the quiver, rather than the thermal, motion and the actual distribution has little effect.

We have carried out extensive simulations of inverse bremsstrahlung absorption using Monte Carlo methods. As the calculation is essentially within the impact approximation and uses multiple small angle scattering described by (1), the calculations examine the validity of the Maxwellian distribution condition. A typical result is shown in Fig. 5 for the mean electron energy gain from radiation of wavelength 1.06 μm in a plasma with $Z=1$ at an electron density of 10^{20} cm^{-3} with an initially thermal distribution at 10 eV temperature. It is easily shown that in this case the range of intensities considered ($10^{12}-10^{17}$ W/cm^2) covers the range $x \ll 1$ to $x \gg 1$. Despite using relatively few electrons (1000) and neglecting electron-electron collisions, the results for the electron energy after 0.4 ps are in good agreement with the formulas proposed over the intensity range. Since the electron scattering interval was $\frac{1}{32}$ of the laser period, each electron underwent about 3500 scatterings, giving a reasonable statistical averaging—checks with additional electrons and collisions showed that the Monte Carlo results were accurate. The results show very much the expected pattern. At high fields ($x \gtrsim 1$) the approximations are reasonably accurate, but at low fields ($x \lesssim 1$) the modification of the distribution as the electrons heat

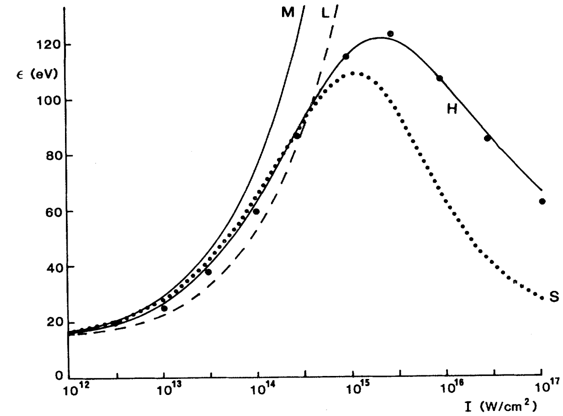

FIG. 5. Comparison of various forms of the absorption rate, low field Maxwellian (M) and Langdon (L), Schlessinger and Wright (S), and high field (H) heating a 10 eV temperature plasma of $Z=1$ at $n_e=10^{20}$ cm^{-3}, with direct Monte Carlo simulation of electron-ion collisions only, at constant intensity I, for 0.4 psec.

leads to a reduction of the absorbed energy, though not to quite such a low value as predicted by the standard Langdon form for reasons discussed earlier. Under these conditions the Langdon formula is generally found to be quite accurate once the distribution has settled into its stationary form provided the field is not strong. The standard high field forms, e.g., that due to Schlessinger and Wright [18], tend to underestimate the absorption at high fields. This can be ascribed to two causes. First, the approximation used does not contain the additional ln(x) term in the asymptotic form of $\mathcal{S}_1(x)$. Second, the Coulomb ln used is based on Eq. (1) and the thermal speed v_T alone. The additional term $\mathcal{S}_2(x)$ in the result associated with the variance of the total speed v in the cycle is neglected. The second is the larger contribution and raises the question of its validity, but it is justified on three grounds: it appears naturally from the asymptotic forms that give rise to the impact approximation [4,5]; it is essential to properly include this term to ensure a nonzero absorption in a Coulomb field at low intensity [1,2,5]; the term associated with the $\mathcal{S}_2(x)$ gives additional terms (-3γ) or (-2γ) for classical or quantal collisions, respectively, as $x \rightarrow 0$. These terms appear naturally in the accurate evaluations for low field single-photon absorption [12].

As a second example we examine the problem considered by Rae and Burnett [19] involving the heating rate of a plasma with $Z=8$ at an electron density 10^{20} cm^{-3} and temperature 30 eV with 0.248 μm wavelength radiation. This is an interesting case as at low fields where the total velocity is near thermal, the condition $b_{\max} \gg b_{\min}$ is not fulfilled and the standard Coulomb ln would be negative. Rae and Burnett [19] appear to have circumvented the problem by using the dc outer cutoff $b_{\max}=\lambda_D$. In fact the Gaunt factor correction is required. Figure 6 shows the calculated heating rate calculated with our form and with Rae and Burnett's results using Schlessinger and Wright's correction. At low in-

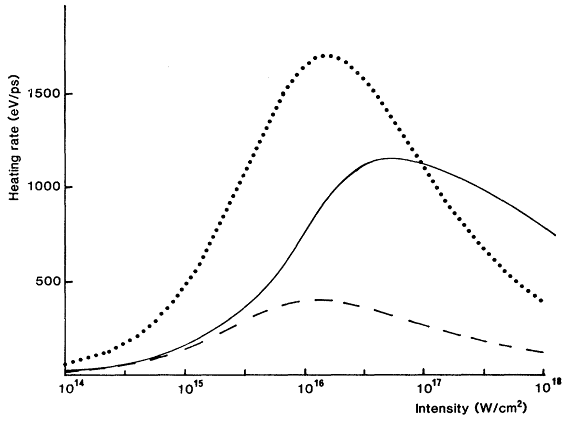

FIG. 6. Heating rate in a plasma with $Z=8$ at ion density 1.25×10^{20} cm^{-3} and low wavelength 0.248 μm as a function of intensity. The heating rate (——) is compared with Schlessinger and Wright's formula with the dc Coulomb logarithm (. . . .) and with the correct form (— — —).

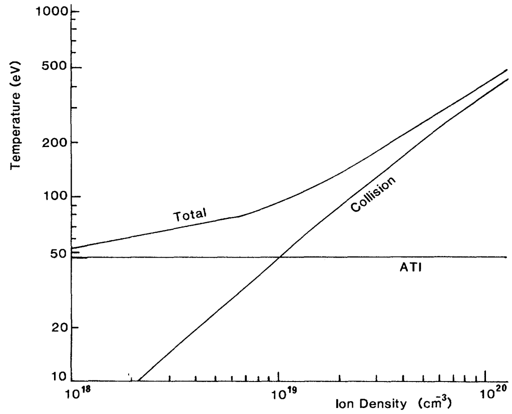

FIG. 8. Dependence of electron temperature on ion density for neon gas irradiated by a 100 ps, 2.5×10^{17} W/cm^2 pulse of wavelength 0.248 μm.

tensity they significantly overestimate the heating due to the incorrect logarithm. However, at high intensity the neglected terms in the correction become important and the results are underestimated.

In order to assess the model for the specific problem of ATI breakdown, we have investigated the breakdown of neon by a laser pulse of full width half maximum duration 350 ps at wavelength 0.25 μm at intensity 1.5×10^{18} W/cm^2 for various densities, assuming that all collisions are classical (Fig. 7). The ATI model is used in the tunneling approximation of Ammosov, Delone, and Krainov [20] with an energy distribution given by Delone and Krainov [21] or, equivalently, by Burnett and Rae [19]. Both electron-ion and electron-electron collisions are treated by a multiple scattering probability distributions (equivalent to Fokker-Planck), similar (but corrected) to that of Takizuka and Abe [22]. The electron-electron collisions are cut off at the Debye length, but the electron-ion collisions at v/ω. The calculations were performed with 100 000 electrons. The agreement between the results is good. In this case the initial electron velocity following ionization is in the field direction and tends to compensate the transverse dominance associated with absorption to produce a nearly isotropic distribution. These results may be considered in relation to the work of Penetrante and Bardsley [23]. In that work the ATI energy at an intensity 10^{18} W/cm^2 in neon using 0.248 μm radiation is calculated at about 80 eV, whereas we estimate a value of about 36 eV, consistent with those quoted by Rae and Burnett [19]. This discrepancy is difficult to understand, but may arise from differences in the pulse shape, as the ATI energy is sensitive to the initial rate of rise of the pulse.

The role of heating in ATI experiments was investigat-

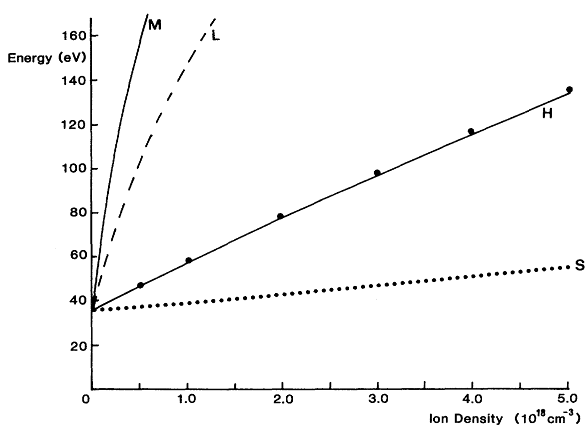

FIG. 7. Calculation of the electron energy ε as a function of ion density n_i in the breakdown of neon by a pulse of 350 fs duration and 1.5×10^{17} W/cm^2 peak intensity, of 0.254 μm wavelength is plotted. The results of direct Monte Carlo calculations with electron relaxation are shown. The curves are calculated for low field Maxwellian (M) and Langdon (L) and high field (H) absorption rates.

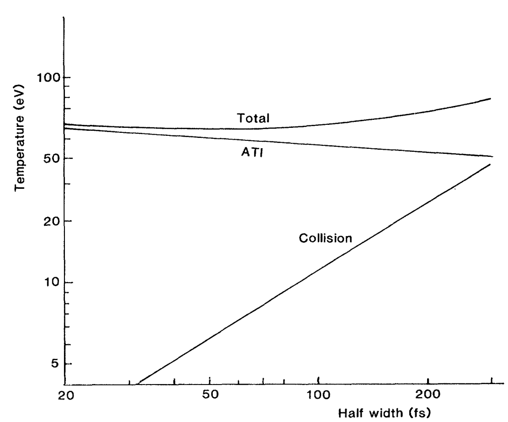

FIG. 9. Dependence of electron temperature on duration of a 2.5×10^{17} W/cm^2 pulse of 0.248 μm wavelength in neon at a density of 2.5×10^{18} cm^{-3}.

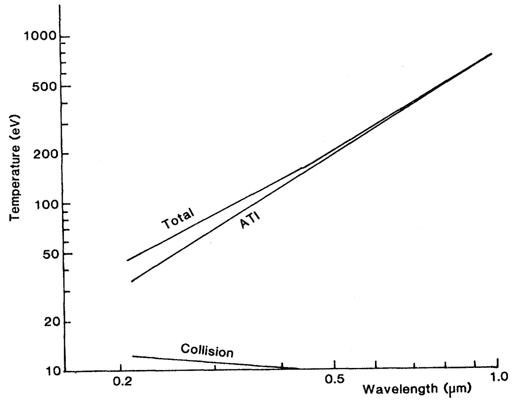

FIG. 10. Dependence of electron temperature on wavelength for a pulse of 100 ps duration, 3.5×10^{17} W/cm^2 intensity at a neon density 2.5×10^{18} cm^{-3}.

ed by Rae and Burnett [19] using a model similar to the analytic form used here, but with an unsatisfactory expression for the collisional absorption as noted earlier. In view of their conclusions regarding the marginal viability of prospective x-ray laser schemes, we have repeated their calculations. The results are shown in Figs. 8, 9, and 10 giving the scaling of electron energy with density, intensity, and laser wavelength, respectively. The results, despite the error, are remarkably similar and their conclusions therefore remain valid. The reason for the agreement is interesting. For short wavelengths ATI heating is small and collisional heating significant. However, in this case the full correction $\mathcal{S}_2(x)$ must be used and gives an effective result very similar to that used by Rae and Burnett. At longer wavelengths, the high field falloff and increased ATI energy renders collisions relatively unimportant.

Finally, we note that a number of the formulas derived here have previously appeared. Silin [24] devised the basic form of (11) for classical collisions and Brysk [29] for quantum collisions, but without taking into account the variation in the Coulomb logarithm. The importance of this term was noted by the present author and included as an additional term to the usually dominant one. Expressions equivalent to (48), (49), and (60) have recently been obtained within the Born approximation by Polishchuk and Meyer-Ter-Vehn [26].

APPENDIX: THE ASYMPTOTIC LIMIT

Consider first the terms

$$T_n(x)=\sqrt{\pi}x^{(n-1/2)}e^{-x}M(\tfrac{1}{2},(n+1),x)/n! \tag{A1}$$

for large x. The hypergeometric function

$$M(\tfrac{1}{2},(n+1),x)=\frac{\Gamma(n+1)}{\Gamma(1/2)}\sum_{m=0}\frac{\Gamma(m+1/2)}{\Gamma(m+n+1)}\frac{x^m}{m!} \tag{A2}$$

$$\simeq\frac{1}{\sqrt{2\pi}}\frac{\Gamma(n+1)}{\Gamma(1/2)}e^{(n+1)}\sum_{m=0}\frac{(m+n+1)^{1/2}}{(m+1)^{1/2}}\exp\{m+m\ln x-(m+n+1)\ln(m+n+1)\} \tag{A3}$$

using Stirling's theorem for the gamma functions. The argument of the exponential has a maximum if $m=[x-(n+1)]$. Hence, replacing the sum by an integration performed by the method of steepest descent,

$$M(\tfrac{1}{2},(n+1)x)\simeq\frac{\Gamma(n+1)}{\Gamma(1/2)}\frac{x^{-(n+1/2)}e^{x}}{(1-n/x)^{1/2}},\quad n<x\ , \tag{A4}$$

For $n>x$ the exponential in (A3) are decaying and give little contribution. Hence we obtain the result quoted earlier (16).

From Eq. (18) the general expression for S is

$$S=\sum_{n=1}[T_n(x)/(2n+1)]\int_0^\infty\ln(1+\bar{\Delta}t^3)t^{(n-1)}dt/(n-1)!\ .$$

For x large it follows from (A4) that the bulk of the contribution to the sum is from the terms with n large. Hence we may perform the integral by the method of steepest descent and again replace the sum by an integral

$$S\simeq\int_1^\infty\frac{\ln(1+\bar{\Delta}n^3)dn}{(2n+1)(1-n/x)^{1/2}}\ ; \tag{A5}$$

substituting $n=x\sin^2\theta$ we obtain (25). The sources of the error discussed in the text can clearly be seen in this calculation arising from neglect of higher-order terms in the replacement of sums by integrals and the improper treatment of initial terms in the sums.

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